

A REGULARITY CRITERION FOR THE 3D FULL COMPRESSIBLE MAGNETOHYDRODYNAMIC EQUATIONS WITH ZERO HEAT CONDUCTIVITY

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ABSTRACT. We establish a regularity criterion for the 3D full compressible magnetohydrodynamic equations with zero heat conductivity and vacuum in a bounded domain.

1. INTRODUCTION

In this paper, we consider the 3D full compressible magnetohydrodynamic equations in a bounded domain $\Omega \subset \mathbb{R}^3$:

$$\partial_t \rho + \operatorname{div}(\rho u) = 0 \quad (1.1)$$

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p - \mu \Delta u - (\lambda + \mu) \nabla \operatorname{div} u = \operatorname{rot} b \times b, \quad (1.2)$$

$$C_V [\partial_t(\rho \theta) + \operatorname{div}(\rho u \theta)] - \kappa \Delta \theta + p \operatorname{div} u = \frac{\mu}{2} |\nabla u + \nabla u^t|^2 + \lambda (\operatorname{div} u)^2 + \nu |\operatorname{rot} b|^2, \quad (1.3)$$

$$\partial_t b + \operatorname{rot}(b \times u) = \nu \Delta b, \quad \operatorname{div} b = 0, \quad (1.4)$$

with the initial and boundary conditions

$$u = 0, \quad \kappa \frac{\partial \theta}{\partial n} = 0, \quad b \cdot n = 0, \quad \operatorname{rot} b \times n = 0 \quad \text{on } \partial \Omega \times (0, \infty), \quad (1.5)$$

$$(\rho, u, \theta, b)(\cdot, 0) = (\rho_0, u_0, \theta_0, b_0) \quad \text{in } \Omega \subset \mathbb{R}^3. \quad (1.6)$$

Here the unknowns ρ, u, p, θ , and b stand for the density, velocity, pressure, temperature, and magnetic field, respectively. The physical constants μ and λ are the shear viscosity and bulk viscosity of the fluid and satisfy $\mu > 0$ and $\lambda + \frac{2}{3}\mu \geq 0$. $C_V > 0$ is the specific heat at constant volume and $\kappa > 0$ is the heat conductivity. $\nu > 0$ is the magnetic diffusivity. ∇u^t denotes the transpose of the matrix ∇u . We assume that Ω is a bounded and simply connected domain in \mathbb{R}^3 with smooth boundary $\partial \Omega$. We use n to denote the outward unit normal vector to $\partial \Omega$.

The full compressible magnetohydrodynamic equations (1.1)-(1.4) can be rigorous derivation from the compressible Navier-Stokes-Maxwell system [14]. Due to

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the physical importance of the magnetohydrodynamics, there are a lot of literature on the system (1.1)-(1.4), among others, we mention [8] on the local strong solutions, [4, 9, 10] on the global weak solutions, [15, 16] on low Mach number limit, and [19] on the time decay of smooth small solutions.

Assume that the pressure take the form $p = R\rho\theta$ with R being the generic gas constant.

In [11], Huang and Li proved the following regularity criterion

$$\rho \in L^\infty(0, T; L^\infty) \text{ and } u \in L^s(0, T; L^r) \text{ with } \frac{2}{s} + \frac{3}{r} = 1 \text{ and } 3 < r \leq \infty, \quad (1.7)$$

with b satisfying the homogeneous Dirichlet boundary condition $b = 0$ on $\partial\Omega \times (0, \infty)$. Later this result was generalized in [7] to the case of the boundary condition (1.5), i.e.,

$$b \cdot n = 0, \quad \text{rot } b \times n = 0 \text{ on } \partial\Omega \times (0, \infty). \quad (1.8)$$

When considering the system (1.1)-(1.4) in a two dimensional domain, Lu, Chen and Huang [18] showed the following regularity criterion

$$\text{div } u \in L^1(0, T; L^\infty) \quad (1.9)$$

with b satisfying the boundary condition $b = 0$ on $\partial\Omega \times (0, \infty)$. Here we remark that same result can be proved for b satisfying the boundary condition: $b \cdot n = 0, \text{rot } b = 0$ on $\partial\Omega \times (0, \infty)$. An related weak result was obtained in [6].

Very recently, Huang and Wang [12] establish the following regularity criterion

$$\rho, \theta, b \in L^\infty(0, T; L^\infty) \text{ with } 2\mu > \lambda. \quad (1.10)$$

for the system (1.1)-(1.4) in the whole space \mathbb{R}^3 with $\kappa = \nu = 0$.

The aim of this paper is to show that the regularity criterion (1.10) still hold for the system (1.1)-(1.4) in a bounded domain with the boundary condition (1.5) when $\kappa = 0$ and $\nu = 1$. We will prove

Theorem 1.1. *Let $\kappa = 0$ and $\nu = 1$. For $q \in (3, 6]$, assume that the initial data $(\rho_0 \geq 0, u_0, p_0 = R\rho_0\theta_0 \geq 0, b_0)$ satisfy*

$$\begin{cases} \rho_0, p_0 \in W^{1,q}(\Omega), \quad u_0 \in H_0^1(\Omega) \cap H^2(\Omega), \quad b_0 \in H^2 \text{ with } \text{div } b_0 = 0 \text{ in } \Omega, \\ b_0 \cdot n = 0, \text{rot } b_0 \times n = 0 \text{ on } \partial\Omega \end{cases} \quad (1.11)$$

and the compatibility condition

$$-\mu\Delta u_0 - (\lambda + \mu)\nabla \text{div } u_0 + \nabla p_0 - \text{rot } b_0 \times b_0 = \sqrt{\rho_0}g, \quad (1.12)$$

with $g \in L^2(\Omega)$. Let (ρ, u, p, b) be a local strong solution to the problem (1.1)-(1.6). If (1.10) holds true with $0 < T < \infty$, then the solution (ρ, u, p, b) can be extended beyond $T > 0$.

We mention that when taking $b = 0$ in the system (1.1)-(1.4), it is reduced to the full compressible Navier-Stokes system and a lot of regularity criteria can be found in [5, 20, 23] and the references cited therein.

The remainder of this paper is devoted to the proof of Theorem 1.1. We give some preliminaries in section 2 and present the proof of Theorem 1.1 in section 3. Below we shall use the letter C to denote the positive constant which may change from line to line.

2. PRELIMINARIES

First, we consider the boundary value problem for the Lamé operator L

$$\begin{cases} LU \triangleq \mu \Delta U + (\mu + \lambda) \nabla \operatorname{div} U = F & \text{in } \Omega, \\ U(x) = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.1)$$

Here $U = (U_1, U_2, U_3)$, $F = (F_1, F_2, F_3)$. It is well known that the system (2.1) is a strongly elliptic system, thus there exists a unique weak solution $U \in H_0^1(\Omega)$ for $F \in W^{-1,2}(\Omega)$.

Lemma 2.1. *Let $q \in (1, \infty)$ and U be a solution of (2.1). There exists a constant C depending only on λ, μ, q and Ω such that the following estimates hold:*

(1) *if $F \in L^q(\Omega)$, then*

$$\|U\|_{W^{2,q}(\Omega)} \leq C \|F\|_{L^q(\Omega)}; \quad (2.2)$$

(2) *if $F \in W^{-1,q}(\Omega)$ (that is, $F = \operatorname{div} f$ with $f = (f_{ij})_{3 \times 3}$, $f_{ij} \in L^q(\Omega)$), then*

$$\|U\|_{W^{1,q}(\Omega)} \leq C \|f\|_{L^q(\Omega)}; \quad (2.3)$$

(3) *if $F = \operatorname{div} f$ with $f_{ij} = \partial_k h_{ij}^k$ and $h_{ij}^k \in W_0^{1,q}(\Omega)$ for $i, j, k = 1, 2, 3$, then*

$$\|U\|_{L^q(\Omega)} \leq C \|h\|_{L^q(\Omega)}. \quad (2.4)$$

Proof. The estimates (2.2) and (2.3) are classical for strongly elliptic systems, see for example [2]. The estimate (2.4) can be proved by a duality argument with the help of (2.2). \square

We need an endpoint estimate for L in the case $q = \infty$. Let $BMO(\Omega)$ stand for the John-Nirenberg space of bounded mean oscillation whose norm is defined by

$$\|f\|_{BMO(\Omega)} := \|f\|_{L^2(\Omega)} + [f]_{BMO},$$

with

$$\begin{aligned} [f]_{BMO(\Omega)} &:= \sup_{x \in \Omega, r \in (0, d)} \frac{1}{|\Omega_r(x)|} \int_{\Omega_r(x)} |f(y) - f_{\Omega_r(x)}| dy, \\ f_{\Omega_r(x)} &:= \frac{1}{|\Omega_r(x)|} \int_{\Omega_r(x)} f(y) dy. \end{aligned}$$

Here $\Omega_r(x) := B_r(x) \cap \Omega$, $B_r(x)$ is a ball with center x and radius r , d is the diameter of Ω and $|\Omega_r(x)|$ denotes the Lebesgue measure of $\Omega_r(x)$.

Lemma 2.2 ([1]). *If $F = \operatorname{div} f$ with $f = (f_{ij})_{3 \times 3}$, $f_{ij} \in L^\infty(\Omega) \cap L^2(\Omega)$, then $\nabla U \in BMO(\Omega)$ and there exists a constant C depending only on λ, μ and Ω such that*

$$\|\nabla U\|_{BMO(\Omega)} \leq C(\|f\|_{L^\infty(\Omega)} + \|f\|_{L^2(\Omega)}). \quad (2.5)$$

Let us conclude this section by recalling a variant of the Brezis-Wagner inequality [3].

Lemma 2.3 ([21]). *Let Ω be a bounded Lipschitz domain in \mathbb{R}^3 and $f \in W^{1,q}$ with $3 < q < \infty$. There exists a constant C depending on q and the Lipschitz property of Ω such that*

$$\|f\|_{L^\infty(\Omega)} \leq C(1 + \|f\|_{BMO(\Omega)} \ln(e + \|\nabla f\|_{L^q(\Omega)})). \quad (2.6)$$

Lemma 2.4 ([13]). *Let b be a solution to the Poisson equation*

$$-\Delta b = f \quad \text{in } \Omega$$

with the boundary condition

$$b \cdot n = 0, \operatorname{rot} b \times n = 0 \quad \text{on } \partial\Omega.$$

Then there holds

$$\|\nabla^2 b\|_{L^p} \leq C\|f\|_{L^p} + C\|\nabla b\|_{L^2} \quad \text{with } 1 < p < \infty. \quad (2.7)$$

In the following proofs, we will use the Poincaré inequality [17]:

$$\|b\|_{L^2} \leq C(\|\operatorname{div} b\|_{L^2} + \|\operatorname{rot} b\|_{L^2}) \quad (2.8)$$

for any $b \in H^1(\Omega)$ with $b \cdot n = 0$ or $b \times n = 0$ on $\partial\Omega$.

We will also use the inequality [22]:

$$\|\nabla b\|_{L^2} \leq C(\|\operatorname{div} b\|_{L^2} + \|\operatorname{rot} b\|_{L^2}) \quad (2.9)$$

for any $b \in H^1(\Omega)$ with $b \cdot n = 0$ or $b \times n = 0$ on $\partial\Omega$.

3. PROOF OF THEOREM 1.1

This section is devoted to the proof of Theorem 1.1, we only need to show a priori estimates. For simplicity, we will take $\nu = C_V = \mathcal{R} = 1$.

Testing (1.2) by u , (1.4) by b , summing up the results and using (1.1) and (1.10), we see that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int (\rho|u|^2 + |b|^2) dx + \int (\mu|\nabla u|^2 + (\lambda + \mu)(\operatorname{div} u)^2 + |\operatorname{rot} b|^2) dx \\ &= - \int u \nabla p dx = \int p \operatorname{div} u dx \leq \frac{\lambda + \mu}{2} \int (\operatorname{div} u)^2 dx + C, \end{aligned}$$

which gives

$$\int (\rho|u|^2 + |b|^2) dx + \int_0^T \int (|\nabla u|^2 + |\operatorname{rot} b|^2) dx dt \leq C. \quad (3.1)$$

Integrating (1.3) over $\Omega \times (0, t)$ and using (1.10) and (3.1), we find that

$$\int \rho \theta dx \leq C. \quad (3.2)$$

By the same calculations as that in [12], we get

$$\int \rho |u|^4 dx + \int_0^T \int |\nabla u|^2 |u|^2 dx dt \leq C. \quad (3.3)$$

We define $v \in H_0^1$ satisfying

$$Lv := \mu \Delta v + (\lambda + \mu) \nabla \operatorname{div} v = \nabla p, \quad (3.4)$$

and $w := u - v$. Thanks to Lemma 2.1, for any $1 < r < \infty$, there hold

$$\|\nabla v\|_{L^r(\Omega)} \leq C \|p\|_{L^r(\Omega)}, \|\nabla^2 v\|_{L^r(\Omega)} \leq C \|\nabla p\|_{L^r(\Omega)}. \quad (3.5)$$

It is easy to see that w satisfies

$$\mu \Delta w + (\lambda + \mu) \nabla \operatorname{div} w = \rho \dot{u} - \operatorname{rot} b \times b, \quad (3.6)$$

Then it follows from Lemma 2.1 that

$$\|\nabla^2 w\|_{L^2(\Omega)} \leq C \|\rho \dot{u}\|_{L^2(\Omega)} + C \|\operatorname{rot} b \times b\|_{L^2(\Omega)}. \quad (3.7)$$

Let E be the specific energy defined by

$$E := \theta + \frac{|u|^2}{2}.$$

Then

$$\begin{aligned} & \partial_t \left(\rho E + \frac{|b|^2}{2} \right) + \operatorname{div} (\rho E u + p u + |b|^2 u) \\ &= \frac{1}{2} \mu \Delta |u|^2 + \mu \operatorname{div} (u \cdot \nabla u) + \lambda \operatorname{div} (u \operatorname{div} u) + \operatorname{div} ((u \cdot b) b) - \operatorname{div} (\operatorname{rot} b \times b). \end{aligned} \quad (3.8)$$

Testing (1.2) by u_t and using (1.1) and denoting $\dot{f} := f_t + u \cdot \nabla f$, we deduce that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int (\mu |\nabla u|^2 + (\lambda + \mu) (\operatorname{div} u)^2) dx + \int \rho |\dot{u}|^2 dx \\ &= \int \rho \dot{u} \cdot (u \cdot \nabla) u dx + \int \left[\left(p + \frac{1}{2} |b|^2 \right) \mathbb{I}_3 - (b \otimes b) \right] : \nabla u_t dx \\ &\leq \frac{1}{8} \int \rho |\dot{u}|^2 dx + C \int |u|^2 |\nabla u|^2 dx + \frac{d}{dt} \int \left[\left(p + \frac{1}{2} |b|^2 \right) \mathbb{I}_3 - (b \otimes b) \right] : \nabla u dx \\ &\quad - \int p_t \operatorname{div} u dx - \int \left[\frac{1}{2} |b|^2 \mathbb{I}_3 - (b \otimes b) \right]_t : \nabla u dx. \end{aligned} \quad (3.9)$$

We remark that

$$\begin{aligned} & - \int p_t \operatorname{div} v dx = \int v \nabla p_t dx = \int v (\mu \Delta v_t + (\lambda + \mu) \nabla \operatorname{div} v_t) dx \\ &= - \frac{1}{2} \frac{d}{dt} \int (\mu |\nabla v|^2 + (\lambda + \mu) (\operatorname{div} v)^2) dx. \end{aligned} \quad (3.10)$$

And according to (3.8) and (1.1),

$$\begin{aligned}
& - \int p_t \operatorname{div} w dx \\
& = - \int \left(\rho E + \frac{1}{2} |b|^2 \right)_t \operatorname{div} w dx + \int \left(\frac{1}{2} \rho |u|^2 \right)_t \operatorname{div} w dx + \int (b \cdot b_t) \operatorname{div} w dx \\
& = - \int \left(\rho E u + p u + |b|^2 u - \frac{1}{2} \mu \nabla |u|^2 - \mu (u \cdot \nabla) u - \lambda u \operatorname{div} u - (u \cdot b) b + \operatorname{rot} b \times b \right) \nabla \operatorname{div} w dx \\
& \quad - \frac{1}{2} \int \operatorname{div} (\rho u) |u|^2 \operatorname{div} w dx + \int \rho u_t u \operatorname{div} w dx + \int b b_t \operatorname{div} w dx \\
& = - \int \left(2 \rho \theta u + |b|^2 u - \frac{1}{2} \mu \nabla |u|^2 - \mu (u \cdot \nabla) u - \lambda u \operatorname{div} u - (u \cdot b) b + \operatorname{rot} b \times b \right) \nabla \operatorname{div} w dx \\
& \quad + \int \rho i u \operatorname{div} w dx + \int b b_t \operatorname{div} w dx \\
& \leq C \int \rho |u|^2 dx + C \|u\|_{L^6}^2 + C \int |u|^2 |\nabla u|^2 dx + C \|\nabla b\|_{L^2}^2 + \delta_1 \|\nabla^2 w\|_{L^2}^2 \\
& \quad + \delta_2 \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + C \|\sqrt{\rho} u\|_{L^4}^2 \|\nabla w\|_{L^4}^2 + C \|\nabla w\|_{L^2}^2 + \delta_3 \|b_t\|_{L^2}^2 \\
& \leq C + C \|\nabla u\|_{L^2}^2 + C \int |u|^2 |\nabla u|^2 dx + C \|\nabla b\|_{L^2}^2 + C \delta_1 \|\sqrt{\rho} \dot{u}\|_{L^2}^2 \\
& \quad + \delta_2 \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + C \|\nabla w\|_{L^2}^{\frac{1}{2}} \|\nabla^2 w\|_{L^2}^{\frac{3}{2}} + C \|\nabla w\|_{L^2}^2 + \delta_3 \|b_t\|_{L^2}^2 \\
& \leq C + C \|\nabla u\|_{L^2}^2 + C \int |u|^2 |\nabla u|^2 dx + C \|\nabla b\|_{L^2}^2 \\
& \quad + C \delta_1 \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + \delta_2 \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + \delta_3 \|b_t\|_{L^2}^2 \tag{3.11}
\end{aligned}$$

for any small $0 < \delta_1, \delta_2$ and δ_3 . Here we have used the Gagliardo-Nirenberg inequality

$$\|\nabla w\|_{L^4} \leq C \|\nabla w\|_{L^2}^{\frac{1}{4}} \|\nabla^2 w\|_{L^2}^{\frac{3}{4}} + C \|\nabla w\|_{L^2}$$

and

$$\|\nabla w\|_{L^2} \leq \|\nabla u\|_{L^2} + \|\nabla v\|_{L^2} \leq C + \|\nabla u\|_{L^2}.$$

Observing that the last term of (3.9) can be bounded as

$$- \int \left[\frac{1}{2} |b|^2 \mathbb{I}_3 - (b \otimes b) \right]_t : \nabla u dx \leq \delta_3 \|b_t\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2. \tag{3.12}$$

On the other hand, testing (1.4) by $b_t - \Delta b$, we get

$$\begin{aligned}
& \frac{d}{dt} \int |\operatorname{rot} b|^2 dx + \int (|b_t|^2 + |\Delta b|^2) dx \\
& = \int |\operatorname{rot} (b \times u)|^2 dx \leq C \int |\nabla u|^2 dx + C \|u\|_{L^6}^2 \|\nabla b\|_{L^3}^2 \\
& \leq C \|\nabla u\|_{L^2}^2 + C \|u\|_{L^6}^2 \|\nabla b\|_{L^2}^2 + \frac{1}{2} \|\Delta b\|_{L^2}^2 + C \|u\|_{L^6}^4 \|\nabla b\|_{L^2}^2. \tag{3.13}
\end{aligned}$$

Here we have used the inequality

$$\|\nabla^2 b\|_{L^2} \leq C \|\Delta b\|_{L^2} + C \|\nabla b\|_{L^2} \tag{3.14}$$

and the Gagliardo-Nirenberg inequality

$$\|\nabla b\|_{L^3}^2 \leq C\|\nabla b\|_{L^2}\|\nabla^2 b\|_{L^2} + C\|\nabla b\|_{L^2}^2. \quad (3.15)$$

Inserting (3.10), (3.11) and (3.12) into (3.9) and combining (3.13) and choosing δ_1, δ_2 and δ_3 suitably small and using the Gronwall inequality, we have

$$\sup_{0 \leq t \leq T} \int (|\nabla u|^2 + |\nabla b|^2) dx + \int_0^T \int (|\sqrt{\rho} u_t|^2 + |b_t|^2 + |\nabla^2 b|^2) dx dt \leq C. \quad (3.16)$$

Now we are in a position to give a high order regularity estimates of the solutions. The calculations were motivated by [20]. First of all, we rewrite the equation (1.2) as

$$\rho \dot{u} + \nabla p - Lu = g := \text{rot } b \times b$$

to find that

$$\begin{aligned} & \rho \dot{u}_t + \rho u \cdot \nabla \dot{u} + \nabla p_t + \text{div}(\nabla p \otimes u) \\ &= \mu[\Delta u_t + \text{div}(\Delta u \otimes u)] + (\lambda + \mu)[\nabla \text{div } u_t + \text{div}(\nabla \text{div } u \otimes u)] + g_t + \text{div}(g \otimes u). \end{aligned}$$

Testing the above equation by \dot{u} and using (1.1), we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int \rho |\dot{u}|^2 dx - \mu \int \dot{u} [\Delta u_t + \text{div}(\Delta u \otimes u)] dx \\ & - (\lambda + \mu) \int \dot{u} [\nabla \text{div } u_t + \text{div}(\nabla \text{div } u \otimes u)] dx \\ &= \int (p_t \text{div } \dot{u} + (u \cdot \nabla) \dot{u} \cdot \nabla p) dx + \int (g_t + \text{div}(g \otimes u)) \dot{u} dx. \end{aligned} \quad (3.17)$$

As in [20], one can estimate the second and third terms in above equation as follows.

$$- \int \dot{u} [\Delta u_t + \text{div}(\Delta u \otimes u)] dx \geq \int \left(\frac{3}{4} |\nabla \dot{u}|^2 - C |\nabla u|^4 \right) dx,$$

and

$$- \int \dot{u} [\nabla \text{div } u_t + \text{div}(\nabla \text{div } u \otimes u)] dx \geq \int \left(\frac{1}{2} (\text{div } \dot{u})^2 - \frac{1}{8} |\nabla \dot{u}|^2 - C |\nabla u|^4 \right) dx.$$

Since $p := \rho \theta$, we rewrite (1.3) as follows,

$$p_t + \text{div}(pu) + p \text{div } u = \frac{\mu}{2} |\nabla u + \nabla u^t|^2 + \lambda (\text{div } u)^2 + |\text{rot } b|^2. \quad (3.18)$$

Using (3.18), as in [12, 20], one can estimate the fourth term in (3.17) as follows.

$$\int (p_t \text{div } \dot{u} + (u \cdot \nabla) \dot{u} \cdot \nabla p) dx \leq C + C \|\nabla u\|_{L^4}^4 + C \|\nabla b\|_{L^4}^4 + \frac{\mu}{8} \|\nabla \dot{u}\|_{L^2}^2. \quad (3.19)$$

Using $b \cdot \nabla b + b \times \text{rot } b = \frac{1}{2} \nabla |b|^2$, and (3.16), we bound the last term of (3.17) as follows.

$$\begin{aligned} & \int (g_t + \text{div}(g \otimes u)) \dot{u} dx \\ &= \int \left[\text{div} \left(\frac{1}{2} |b|^2 \mathbb{I}_3 - b \otimes b \right) + \text{div}(g \otimes u) \right] \dot{u} dx \end{aligned}$$

$$\begin{aligned}
&= - \int \left(\frac{1}{2} |b|^2 \mathbb{I}_3 - b \otimes b + g \otimes u \right) : \nabla \dot{u} dx \\
&\leq C(\|b\|_{L^4}^2 + \|b\|_{L^\infty} \|\operatorname{rot} b\|_{L^3} \|u\|_{L^6}) \|\nabla \dot{u}\|_{L^2} \\
&\leq C(1 + \|\nabla b\|_{L^3}) \|\nabla \dot{u}\|_{L^2} \leq \frac{\mu}{8} \|\nabla \dot{u}\|_{L^2}^2 + C \|\nabla b\|_{L^3}^2 + C.
\end{aligned}$$

Inserting the those estimates into (3.16) and using

$$\begin{aligned}
\|\nabla b\|_{L^4}^4 &\leq C \|b\|_{L^\infty}^2 \|b\|_{H^2}^2, \\
\|\nabla u\|_{L^4}^4 &\leq \|\nabla u\|_{L^2} \|\nabla u\|_{L^6}^3 \leq C(\|\nabla v\|_{L^6} + \|\nabla w\|_{L^6}^3) \leq C(1 + \|\sqrt{\rho} \dot{u}\|_{L^2}^3),
\end{aligned}$$

We have

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int \rho |\dot{u}|^2 dx + \mu \int |\nabla \dot{u}|^2 dx + (\lambda + \mu) \int (\operatorname{div} \dot{u})^2 dx \\
&\leq C(1 + \|\nabla u\|_{L^4}^4 + \|b\|_{H^2}^2) \\
&\leq C + C \|\sqrt{\rho} \dot{u}\|_{L^2}^4 + C \|b\|_{H^2}^2,
\end{aligned}$$

which gives

$$\|\sqrt{\rho} \dot{u}\|_{L^\infty(0,T;L^2)} + \|\dot{u}\|_{L^2(0,T;H^1)} \leq C. \quad (3.20)$$

By the same calculations as in [12], it is easy to verify that

$$\sup_{0 \leq t \leq T} \|\nabla w\|_{H^1} + \int_0^T (\|\nabla^2 w\|_{L^p}^2 + \|\nabla w\|_{L^\infty}^2) dt \leq C \quad \text{with any } 2 \leq p \leq 6. \quad (3.21)$$

Applying ∂_t to (1.4), testing the result by b_t , using (3.3) and (3.20), we have

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int |b_t|^2 dx + \int |\operatorname{rot} b_t|^2 dx = - \int \partial_t (b \times u) \operatorname{rot} b_t dx \\
&= - \int (b_t \times u + b \times \dot{u} - b \times (u \cdot \nabla) u) \operatorname{rot} b_t dx \\
&\leq (\|b_t\|_{L^3} \|u\|_{L^6} + \|b\|_{L^3} \|\dot{u}\|_{L^6} + \|b\|_{L^\infty} \|u \cdot \nabla u\|_{L^2}) \|\operatorname{rot} b_t\|_{L^2} \\
&\leq C(\|b_t\|_{L^3} + \|\dot{u}\|_{L^6} + \|u \cdot \nabla u\|_{L^2}) \|\operatorname{rot} b_t\|_{L^2} \\
&\leq \frac{1}{2} \|\operatorname{rot} b_t\|_{L^2}^2 + C \|b_t\|_{L^2}^2 + C \|\dot{u}\|_{L^6}^2 + C \|u \cdot \nabla u\|_{L^2}^2,
\end{aligned}$$

which implies

$$\|b_t\|_{L^\infty(0,T;L^2)} + \|b_t\|_{L^2(0,T;H^1)} \leq C. \quad (3.22)$$

This and (1.4) and (3.16) lead to

$$\|b\|_{L^\infty(0,T;H^2)} + \|b\|_{L^2(0,T;W^{2,6})} \leq C, \quad (3.23)$$

where we used

$$\|u\|_{L^\infty(0,T;W^{1,6})} \leq \|v\|_{L^\infty(0,T;W^{1,6})} + \|w\|_{L^\infty(0,T;W^{1,6})} \leq C.$$

Direct calculations show that

$$\frac{d}{dt} \|\nabla \rho\|_{L^q} \leq C(1 + \|\nabla u\|_{L^\infty}) \|\nabla \rho\|_{L^q} + C \|\nabla^2 u\|_{L^q}, \quad (3.24)$$

and

$$\frac{d}{dt}\|\nabla p\|_{L^q} \leq C(1 + \|\nabla u\|_{L^\infty})(\|\nabla p\|_{L^q} + \|\nabla^2 u\|_{L^q}) + C\|\nabla b\|_{L^\infty}\|\nabla^2 b\|_{L^q}. \quad (3.25)$$

We bound the last term of (3.25) as follows.

$$\|\nabla b\|_{L^\infty}\|\nabla^2 b\|_{L^q} \leq C(1 + \|\nabla^2 b\|_{L^q})\|\nabla^2 b\|_{L^q} \leq C + C\|\nabla^2 b\|_{L^q}^2. \quad (3.26)$$

As in [12], it is easy to prove that

$$\|\nabla \rho\|_{L^\infty(0,T;L^q)} + \|\nabla p\|_{L^\infty(0,T;L^q)} \leq C, \quad (3.27)$$

$$\|\nabla u\|_{L^2(0,T;L^\infty)} + \|u\|_{L^\infty(0,T;H^2)} \leq C. \quad (3.28)$$

This completes the proof. \square

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